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APPLICATION OF A BÄCKLUND TRANSFORMATION IN GRAVITY DRIVEN TWO PHASE FILTRATION

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Abstract—A nonlinear boundary value/initial value problem in two-phase filtration under gravity is solved analytically via the application of two successive Bäcklund transformations.

1. INTRODUCTION

A nonlinear boundary value/initial value problem is presented which describes one-dimensional two-phase flow under gravity in a semi-infinite porous medium with zero boundary injection. The problem is reduced to a *linear* boundary value/initial value problem via the application of two successive Bäcklund transformations and thereby solved analytically. It is noted that an analogous boundary value problem without gravitational effects but with constant injection at the reservoir boundary has been recently solved by Fokas & Yortsos (1982).

2. THE MATHEMATICAL MODEL IN TWO-PHASE FILTRATION

If S_w and S_o denote, in turn, water and oil saturations where the saturation S of a phase is defined as the fraction of the void volume occupied by that phase, then

$$S_w + S_o = 1, \tag{1}$$

since the two fluids jointly fill the void in the porous medium.

If q_w and q_o denote the water and oil flow rates respectively per cross-sectional area of the reservoir and ϕ is the porosity (assumed constant) of the reservoir, then the conservation laws for each phase yield

$$\phi \frac{\partial S_w}{\partial \tau} + \frac{\partial q_w}{\partial \xi} = 0, \tag{2}$$

$$\phi \frac{\partial S_o}{\partial \tau} + \frac{\partial q_o}{\partial \xi} = 0, \tag{3}$$

where ξ and τ are dimensional space and time measures.

Darcy's law applied to both phases produces

$$q_w = -\lambda_w \left[\frac{\partial P_w}{\partial \xi} + \rho_w g \right], \tag{4}$$

$$q_0 = -\lambda_0 \left[\frac{\partial P_o}{\partial \xi} + \rho_o g \right], \tag{5}$$

where the phase mobilities λ_w and λ_o are given by

$$\lambda_w = kk_{rw}/\mu_w, \quad [6]$$

$$\lambda_o = kk_{ro}/\mu_0 \quad [7]$$

and ρ_o, $\rho_w(>\rho_o)$ represent the respective densities (assumed constant) of the two phases. In the above, k is the absolute permeability of the reservoir, μ_w, μ_o and P_w, P_o denote the viscosity and pressure of each phase, while k_{rw} and k_{ro} denote the relative permeabilities. The quantities k_{rw} and k_{ro} are functions of their respective saturations. In view of the relation [1], we can select one saturation, here $S = S_o$, to be the single variable on which k_{rw} and k_{ro} depend. The capillary pressure $P_c(S)$ is given in terms of P_w and P_o via the relation

$$P_c = P_o - P_w. \quad [8]$$

In any water–oil reservoir system such as is considered here, the following monotonicity features are apparent (Fokas & Yortsos 1982):

$$\frac{dk_{rw}}{dS} < 0, \frac{dk_{ro}}{dS} > 0, \frac{dP_c}{dS} > 0, \quad [9]$$

while at the endpoints

$$k_{rw} = 1, k_{ro} = 0, \frac{dP_c}{dS} \to \infty \text{ at } S = S_{o,\min}, \quad [10]$$

$$k_{rw} = 0, k_{ro} = 1, \frac{dP_c}{dS} \to \infty \text{ at } S = S_{o,\max} = 1 - S_{w,\min}, \quad [11]$$

where the irreducible saturations $S_{o,\min}$ and $S_{w,\min}$ are constants of the system and denote the least value of the saturation that each phase may adopt.

3. THE NONLINEAR BOUNDARY VALUE/INITIAL VALUE PROBLEM

If there is zero injection at the reservoir boundary $\xi = 0$ of the porous medium occupying the region $\xi < 0$ then at very large and negative ξ it is required that the oil saturation S be bounded above by its initial level (assumed constant) $S = \bar{S} \le S_{o,\max}$ so that the appropriate boundary conditions become

$$\left.\begin{array}{c} q_0(0,\tau) = q_w(0,\tau) = 0, \tau > 0 \\ S \le S \text{ as } \xi \to -\infty, \tau > 0 \end{array}\right\} \quad [12]$$

to which must be adjoined the initial condition

$$S(\xi, 0) = \bar{S}, \xi < 0. \quad [13]$$

Addition of [2] and [3] together with use of [1] and the above boundary condition $[12]_1$ shows that

$$q_w + q_o = 0, \quad [14]$$

while [4] and [5] may then be written as

$$-q_w/\lambda_w = \frac{\partial P_w}{\partial \xi} + \rho_w g, \quad [15]$$

and

$$q_w/\lambda_o = \frac{\partial P_o}{\partial \xi} + \rho_o g. \qquad [16]$$

Subtraction of the latter equations and use of [8] now shows that

$$q_o = -\left(\frac{\lambda_o \lambda_w}{\lambda_o + \lambda_w}\right)\left[\frac{\partial P_c}{\partial \xi} + (\rho_o - \rho_w) g\right] \qquad [17]$$

and insertion of this expression into [3] yields

$$\frac{\partial S}{\partial \tau} = \frac{1}{\phi}\frac{\partial}{\partial \xi}\left[\left(\frac{\lambda_o \lambda_w}{\lambda_o + \lambda_w}\right)\left\{\frac{\partial P_c}{\partial \xi} + (\rho_o - \rho_w) g\right\}\right], \ \xi < 0, \tau > 0. \qquad [18]$$

Thus, it is required to solve the nonlinear equation [18] subject to the boundary and initial conditions [12] and [13] respectively.

4. THE MODEL LAWS

In the case of the straight line relative permeability approximations appropriate for water–oil systems with negligible interfacial tension

$$k_{ro} = S - S_{o,\min}, \qquad [19]$$

$$k_{rw} = 1 - S_{w,\min} - S \qquad [20]$$

so that

$$\frac{\lambda_o \lambda_w}{(\lambda_o + \lambda_w)} = \frac{k k_{ro} k_{rw}}{(\mu_w k_{ro} + \mu_o k_{rw})} = \frac{k(S - S_{or})(1 - S_{wr} - S)}{[\mu_w (S - S_{or}) + \mu_o (1 - S_{wr} - S)]} \qquad [21]$$

where, we have written

$$S_{o,\min} \equiv S_{or}, \ S_{w,\min} \equiv S_{wr}.$$

If $P_c \equiv \Psi(S)$, then [18] becomes

$$\frac{\partial S}{\partial \tau} = \frac{\partial}{\partial \xi}\left[g(S)\frac{\partial S}{\partial \xi} + K(S)\right], \qquad [22]$$

where

$$g(S) \equiv \frac{1}{\phi}\left(\frac{\lambda_o \lambda_w}{\lambda_o + \lambda_w}\right)\Psi'(S) \qquad [23]$$

and

$$K(S) \equiv \frac{\lambda_o \lambda_w}{\phi(\lambda_o + \lambda_w)}(\rho_o - \rho_w) g. \qquad [24]$$

By virtue of [21],

$$K(S)=\left[\frac{-\alpha}{\beta(\beta S+\gamma)}+\delta\right]\frac{k(S-S_{or})(\rho_o-\rho_w)g}{\mu_o}, \qquad [25]$$

where

$$\delta=\frac{\mu_o}{\phi(\mu_o-\mu_w)}, \qquad [26]$$

$$\frac{\alpha}{\beta^2}=-\frac{\mu_o\mu_w(1-S_{or}-S_{wr})}{\phi(\mu_o-\mu_w)^2} \qquad [27]$$

$$\frac{\gamma}{\beta}=\frac{\mu_w S_{or}-\mu_o(1-S_{wr})}{(\mu_o-\mu_w)}. \qquad [28]$$

If we now set

$$\frac{1}{\phi}\left(\frac{\lambda_o\lambda_w}{\lambda_o+\lambda_w}\right)\Psi'(S)=\frac{1}{(\beta S+\gamma)^2}, \qquad [29]$$

then it emerges that not only is reduction of [22] to Burgers' equation with a convective term achieved, but also the resultant two-parameter $P_c(S)$ curves exhibit both the required monotonicity condition[9]$_3$ and the asymptotic properties at $S=S_{or}$ and $S=1-S_{wr}$ given in[10] and [11].

Thus, [29] yields, on use of [21],

$$\frac{1}{\phi}\frac{k(S-S_{or})(1-S_{wr}-S)\Psi'(S)}{[\mu_w(S-S_{or})+\mu_o(1-S_{wr}-S)]}=\frac{1}{\beta^2\left(S+\frac{\gamma}{\beta}\right)^2} \qquad [30]$$

whence,

$$\Psi'(S)=\frac{\phi(\mu_0-\mu_w)}{\beta^2 k(S-S_{or})(S-[1-S_{wr}])\left(S+\frac{\gamma}{\beta}\right)}>0 \qquad [31]$$

since

$$\left.\begin{aligned}\phi, k>0, \mu_o>\mu_w,\\ S>S_{or}, S<1-S_{wr}=S_{o,\max}\end{aligned}\right\} \qquad [32]$$

and

$$S+\frac{\gamma}{\beta}=[\mu_w(S_{o,\min}-S)+\mu_o(S-S_{o,\max})]/(\mu_o-\mu_w)<0. \qquad [33]$$

Further, [31] shows that

$$\lim_{S\to S_{or}^+}\Psi'(S)=\lim_{S\to(1-S_{wr})^-}\Psi'(S)=+\infty \qquad [34]$$

so that both the required monotonicity and asymptotic endpoint conditions are reproduced in the two-parameter $P_c(S)$-curves generated by the relation[30] viz:

$$P_c(S)=\frac{\phi(\mu_o-\mu_w)}{\beta^2 k}\ln\left\{(S-S_{or})^a(S-(1-S_{wr}))^b\left(S+\frac{\gamma}{\beta}\right)^c\right\}+d, \qquad [35]$$

where

$$a=\left[(S_{or}-(1-S_{wr}))\left(S_{or}+\frac{\gamma}{\beta}\right)\right]^{-1},\ b=\left[(1-S_{wr}-S_{or})\left(1-S_{wr}+\frac{\gamma}{\beta}\right)\right]^{-1}$$
$$c=\left[\left(\frac{\gamma}{\beta}+S_{or}\right)\left(\frac{\gamma}{\beta}+(1-S_{wr})\right)\right]^{-1} \qquad [36]$$

and d is an arbitrary constant of integration.

Thus, for the two-parameter $P_c(S)$-curves given by [35], the nonlinear saturation equation [22] adopts the form

$$\frac{\partial S}{\partial \tau}=\frac{\partial}{\partial \xi}\left[\frac{1}{(\beta S+\gamma)^2}\frac{\partial S}{\partial \gamma}+\left\{\frac{-\alpha}{\beta(\beta S+\gamma)}+\delta\right\}\frac{k(S-S_{or})(\rho_o-\rho_w)g}{\mu_o}\right]. \qquad [37]$$

5. REDUCTION TO CANONICAL FORM VIA A BÄCKLUND TRANSFORMATION

In what follows, we make use of the following result which generalises an invariance property given in Rogers & Shadwick (1982):

Theorem

The nonlinear equation

$$\frac{\partial S}{\partial \tau}-\frac{\partial}{\partial \xi}\left\{g(S)\frac{\partial S}{\partial \xi}+K(S)\right\}=0, \qquad [38]$$

is mapped to

$$\frac{\partial S'}{\partial \tau'}-\frac{\partial}{\partial \xi'}\left\{g'(S')\frac{\partial S'}{\partial \xi'}+K'(S')\right\}=0 \qquad [39]$$

under the Bäcklund transformation

$$\left.\begin{aligned}
&\frac{\partial S'}{\partial \xi'}=\frac{-\beta}{(\beta S+\gamma)^3}\frac{\partial S}{\partial \xi},\\
&\frac{\partial S'}{\partial \tau'}=\frac{\beta}{(\beta S+\gamma)^2}\left[-\frac{\partial S}{\partial \tau}+\frac{\beta}{(\beta S+\gamma)}\left\{g(S)\frac{\partial S}{\partial \xi}+K(S)\right\}\frac{\partial S}{\partial \xi}\right],\\
&\mathrm{d}\xi'=(\beta S+\gamma)\,\mathrm{d}\xi+\beta\left[g(S)\frac{\partial S}{\partial \xi}+K(S)\right]\mathrm{d}\tau,\\
&\tau'=\tau,
\end{aligned}\right\} \qquad [40]$$

where

$$\left.\begin{aligned}
&S'=\frac{1}{(\beta S+\gamma)},\ K'(S')=\frac{-\beta K(S)}{(\beta S+\gamma)},\\
&g'(S')=(\beta S+\gamma)^2 g(S),\\
&0<|\beta S+\gamma|<\infty.
\end{aligned}\right\} \qquad [41]$$

Proof

It is seen that

$$S'\,d\xi' + [g'(S')S'_{\xi'} + K'(S)]\,d\tau' = S'[(\beta S+\gamma)\,d\xi + \beta\{g(S)S_\xi + K(S)\}\,d\tau] + \left[\frac{-\beta g'(S')}{(\beta S+\gamma)^3}S_\xi + K'(S')\right]d\tau' = d\xi$$

so that

$$\frac{\partial S'}{\partial \tau'} - \frac{\partial}{\partial \xi'}\left\{g'(S')\frac{\partial S'}{\partial \xi'} + K'(S')\right\} = 0. \qquad \square$$

Now, in the case of the nonlinear saturation equation[37]

$$\begin{aligned} K'(S') &= \frac{-\beta}{(\beta S+\gamma)}\left[\frac{-\alpha}{\beta(\beta S+\gamma)} + \delta\right]\frac{k(S-S_{or})(\rho_o-\rho_w)g}{\mu_o} \\ &= -\beta S'\left[-\frac{\alpha}{\beta}S' + \delta\right]\frac{k(\rho_o-\rho_w)g}{\mu_o}\left\{\frac{1}{\beta S'} - \frac{\gamma}{\beta} - S_{or}\right\} \\ &= \alpha' S'^2 + \beta' S' + \gamma' \end{aligned} \qquad [42]$$

where

$$\alpha' = \frac{-\alpha k(\rho_o-\rho_w)g}{\mu_o}\left(\frac{\gamma}{\beta} + S_{or}\right), \qquad [43]$$

$$\beta' = \frac{k\beta(\rho_o-\rho_w)g}{\mu_o}\left(\frac{\alpha}{\beta^2} + \delta\left(\frac{\gamma}{\beta} + S_{or}\right)\right) \qquad [44]$$

$$\gamma' = \frac{-\delta k(\rho_o-\rho_w)g}{\mu_o}. \qquad [45]$$

Moreover,

$$g'(S') = S'^{-2}g(S) = +1, \qquad [46]$$

so that, under the Bäcklund transformation given by [40]–[41], the nonlinear saturation equation [37] is mapped to Burgers' equation with a convective term, namely,

$$\frac{\partial S'}{\partial \tau'} = \frac{\partial^2 S'}{\partial \xi'^2} + 2\alpha' S'\frac{\partial S'}{\partial \xi'} + \beta'\frac{\partial S'}{\partial \xi'}. \qquad [47]$$

If we now introduce new independent variables ξ^* and τ^* according to

$$\xi^* = \xi' + \beta'\tau', \qquad [48]$$

$$\tau^* = \tau', \qquad [49]$$

then

$$\frac{\partial}{\partial \tau'} - \beta'\frac{\partial}{\partial \xi'} = \frac{\partial}{\partial \tau^*}, \qquad [50]$$

$$\frac{\partial}{\partial \xi'} = \frac{\partial}{\partial \xi^*} \qquad [51]$$

and we obtain reduction of [47] to Burgers' equation, that is,

$$\frac{\partial S'}{\partial \tau^*} = \frac{\partial^2 S'}{\partial \xi^{*2}} + 2\alpha' S' \frac{\partial S'}{\partial \xi^*}. \qquad [52]$$

6. THE NEW BOUNDARY VALUE/INITIAL VALUE PROBLEM

Under the Bäcklund transformation of the preceding section, the boundary value/initial value problem to be solved becomes

$$\left.\begin{aligned} &\frac{\partial S'}{\partial \tau^*} = \frac{\partial^2 S'}{\partial \xi^{*2}} + 2\alpha' S' \frac{\partial S'}{\partial \xi^*}, \tau^* > 0 \\ &S' = \frac{1}{\beta \bar{S} + \gamma} \text{ at } \tau^* = 0 \\ &S' \le \frac{1}{\beta \bar{S} + \gamma} \text{ as } \xi^* \to -\infty\ (\beta < 0) \end{aligned}\right\} \qquad [53]$$

together with the nonlinear boundary condition associated with the zero injection condition $[12]_1$. The latter is now examined in detail.

Thus, $[12]_1$ together with [15]–[16] provide the boundary condition

$$\frac{\partial P_c}{\partial \xi} = (\rho_w - \rho_o) g \text{ on } \xi = 0, \qquad [54]$$

leading, on use of $P_c \equiv \Psi(S)$, to the flux condition

$$\begin{aligned} \frac{\partial S}{\partial \xi} &= \frac{(\rho_w - \rho_o) g}{\Psi'(S)} \\ &= \frac{\beta^2 k g}{\phi(\mu_o - \mu_w)} (\rho_w - \rho_o)(S - S_{or})(S - [1 - S_{wr}]) \left(S + \frac{\gamma}{\beta}\right) \end{aligned} \qquad [55]$$

on

$$\xi = 0.$$

Under the Bäcklund transformation, the boundary condition [55] becomes

$$\begin{aligned} \frac{\partial S'}{\partial \xi'} &= \frac{k[1 - S_{wr} - S](\rho_w - \rho_o) g \beta [S' - (\gamma + \beta S_{or}) S'^2]}{\phi(\mu_o - \mu_w)} \\ &= \frac{k(\rho_w - \rho_o) g \beta}{\phi(\mu_o - \mu_w)} \left[-\left(1 - S_{wr} + \frac{\gamma}{\beta}\right)(\gamma + \beta S_{or}) S'^2 \right. \\ &\quad \left. + \left\{\left(\frac{\gamma}{\beta} + S_{or}\right) + \left(1 - S_{wr} + \frac{\gamma}{\beta}\right)\right\} S' - \frac{1}{\beta}\right] \end{aligned} \qquad [56]$$

on

$$\begin{aligned} \mathrm{d}\xi' &= \beta[g(S) S_\xi + K(S)]\, \mathrm{d}\tau \\ &= -\frac{1}{S'}[S'_\xi + \alpha' S'^2 + \beta' S' + \gamma']\, \mathrm{d}\tau = [\delta^* - \beta']\, \mathrm{d}\tau \end{aligned} \qquad [57]$$

where

$$\delta^* = \frac{\beta k (1 - S_{wr} - S_{or})(\rho_w - \rho_o) g (\mu_o + \mu_w)}{\phi(\mu_o - \mu_w)^2}. \qquad [58]$$

Thus, under the further transformation [48]–[49], the boundary condition becomes

$$\frac{\partial S'}{\partial \xi^*}=\frac{k(\rho_w-\rho_o)g\beta}{\phi(\mu_o-\mu_w)}\left[-\left(1-S_{wr}+\frac{\gamma}{\beta}\right)(\gamma+\beta S_{or})S'^2 + \left\{\left(\frac{\gamma}{\beta}+S_{or}\right)+\left(1-S_{wr}+\frac{\gamma}{\beta}\right)\right\}S'-\frac{1}{\beta}\right] \tag{59}$$

on

$$\mathrm{d}\xi^*=\delta^*\,\mathrm{d}\tau^*. \tag{60}$$

On application of the Cole–Hopf transformation

$$S'=S^*_{\xi^*}/(\alpha' S^*), \tag{61}$$

the preceding problem may be reduced to consideration of the boundary value/initial value problem

$$\left.\begin{aligned}&\frac{\partial S^*}{\partial \tau^*}-\frac{\partial^2 S^*}{\partial \xi^{*2}}=0,\ \tau^*>0\\ &S^*=S^*(0)\,\mathrm{e}^{\alpha'\xi^*/[\beta(\bar{S}+\gamma/\beta)]}\ \text{at}\ \tau^*=0\dagger\\ &S^*\to 0\ \text{as}\ \xi^*\to-\infty,\tau^*>0\ (\beta<0)\end{aligned}\right\} \tag{62}$$

together with

$$\alpha'\frac{S^*_{\tau^*}}{S^*}=\left(\frac{S^*_{\xi^*}}{S^*}\right)^2\left[\alpha'-\frac{k(\rho_w-\rho_o)g\beta\left(1-S_{wr}+\frac{\gamma}{\beta}\right)(\gamma+\beta S_{or})}{\phi(\mu_o-\mu_w)}\right] +\frac{\alpha' k(\rho_w-\rho_o)g\beta}{\phi(\mu_o-\mu_w)}\left(\frac{S^*_{\xi^*}}{S^*}\right)\left[\frac{2\gamma}{\beta}+S_{or}+1-S_{wr}\right]-\frac{\alpha'^2 k(\rho_w-\rho_o)g}{\phi(\mu_o-\mu_w)}, \tag{63}$$

on

$$\mathrm{d}\xi^*=\delta^*\,\mathrm{d}\tau^*.$$

But,

$$\alpha'-\frac{k(\rho_w-\rho_o)g\beta\left(1-S_{wr}+\frac{\gamma}{\beta}\right)(\gamma+\beta S_{or})}{\phi(\mu_o-\mu_w)}=k(\rho_w-\rho_o)g\left(\frac{\gamma}{\beta}+S_{or}\right)\left[\frac{\alpha}{\mu_o}-\beta^2\frac{\left(1-S_{wr}+\frac{\gamma}{\beta}\right)}{\phi(\mu_o-\mu_w)}\right]=0,$$

so that the boundary condition [63] becomes *linear*, that is,

$$S^*_{\tau^*}=-\delta^* S^*_{\xi^*}-\epsilon^* S^*\ \text{on}\ \mathrm{d}\xi^*=\delta^*\,\mathrm{d}\tau^*, \tag{64}$$

†We subsequently set $S^*(0)=1$ without loss of generality.
Moreover, since

$$\alpha<0,\beta<0,S+\frac{\gamma}{\beta}<0$$

it follows that, for $\rho_o<\rho_w$,

$$\frac{\alpha'}{\beta\left(\bar{S}+\frac{\gamma}{\beta}\right)}=-\frac{\alpha k(\rho_o-\rho_w)g\left(S_{or}+\frac{\gamma}{\beta}\right)}{\beta\mu_o\left(\bar{S}+\frac{\gamma}{\beta}\right)}>0.$$

where

$$\epsilon^* = \frac{\alpha' k(\rho_w - \rho_o)g}{\phi(\mu_o - \mu_w)}. \quad [65]$$

SUMMARY

The *nonlinear* boundary value/initial value problem

$$\left.\begin{aligned} &\frac{\partial S}{\partial \tau} = \frac{1}{\phi}\frac{\partial}{\partial \xi}\left[\left(\frac{\lambda_o\lambda_w}{\lambda_o+\lambda_w}\right)\left\{\frac{\partial P_c}{\partial \xi} + (\rho_o - \rho_w)g\right\}\right], \xi < 0, \tau > 0,\\ &S = \bar{S} \text{ at } \tau = 0,\\ &S \le \bar{S} \text{ as } \xi \to -\infty, \tau > 0\\ &\frac{\partial S}{\partial \xi} = \frac{\beta^2 kg(\rho_w - \rho_o)(S - S_{or})(S - [1 - S_{wr}])\left(S + \frac{\gamma}{\beta}\right)}{\phi(\mu_o - \mu_w)}\\ &\text{on } \xi = 0 \end{aligned}\right\} \quad [66]$$

is mapped, under the transformation

$$\begin{aligned} &(\beta S + \gamma)^{-1} = \frac{S^*_{\xi^*}}{\alpha' S^*},\\ &d\xi^* = (\beta S + \gamma)\, d\xi + \left[\frac{\beta}{(\beta S + \gamma)^2} S_\xi + \frac{\beta}{\mu_o}\left\{\frac{-\alpha}{\beta(\beta S + \gamma)} + \delta\right\} k(S - S_{or})(\rho_o - \rho_w)g + \beta'\right] d\tau,\\ &\tau^* = \tau \end{aligned} \quad [67]$$

to the *linear* boundary value/initial value problem

$$\left.\begin{aligned} &\frac{\partial S^*}{\partial \tau^*} = \frac{\partial^2 S^*}{\partial \xi^{*2}}, \tau^* > 0,\\ &S^* = e^{\alpha'\xi^*/[\beta(\bar{S}+\gamma/\beta)]} \text{ at } \tau^* = 0,\\ &S^* \to 0 \text{ as } \xi^* \to -\infty, \tau^* > 0\ (\beta < 0)\\ &\frac{\partial S^*}{\partial \tau^*} + \delta^* \frac{\partial S^*}{\partial \xi^*} + \epsilon^* S^* = 0 \text{ on } \xi^* = \delta^*\tau^* \end{aligned}\right\} \quad [68]$$

where

$$\xi^* = \int_0^\xi (\beta S(\sigma, \tau) + \gamma)\, d\sigma + \delta^*\tau, \quad [69]$$

and the two-parameter $P_c(S)$ laws are as given by [35].

If we now set

$$\zeta = \xi^* - \delta^*\tau^*, \quad [70]$$

$$t = \tau^*, \quad [71]$$

so that

$$\frac{\partial}{\partial \xi^*} = \frac{\partial}{\partial \zeta}, \quad [72]$$

$$\frac{\partial}{\partial \tau^*} = \frac{\partial}{\partial t} - \delta^* \frac{\partial}{\partial \zeta}, \quad [73]$$

then the boundary value/initial value problem becomes

$$\left.\begin{aligned} &\frac{\partial S^*}{\partial t}=\frac{\partial^2 S^*}{\partial \zeta^2}+\delta^*\frac{\partial S^*}{\partial \zeta},\ t>0,\\ &S^*=e^{\alpha'\zeta/[\beta(\bar{S}+\gamma/\beta)]}\text{ at } t=0,\\ &S^*\to 0 \text{ as } \zeta\to-\infty,\ t>0\ (\beta<0),\\ &\frac{\partial S^*}{\partial t}+\epsilon^* S^*=0 \text{ on } \zeta=0. \end{aligned}\right\} \qquad [74]$$

Once this boundary value/initial value problem has been solved for $S^*\equiv\Phi(\zeta, t)$ then $S(\xi,\tau)$ is given parametrically via the relations

$$S=\frac{\alpha'}{\beta}\frac{\Phi}{\Phi_\zeta}-\frac{\gamma}{\beta},\ \xi=\int_0^\zeta\{\Phi_\sigma(\sigma,t)/[\alpha'\Phi]\}\,d\sigma,\ t=\tau. \qquad [75]$$

7. SOLUTION OF THE CANONICAL BOUNDARY VALUE PROBLEM

Application to $[74]_1$ of a Laplace transformation with respect to t yields, on use of the initial condition $[74]_2$,

$$\hat{S}_{\zeta\zeta}+\delta^*\hat{S}_\zeta-s\hat{S}=-e^{\alpha'\zeta/[\beta(\bar{S}+\gamma/\beta)]}, \qquad [76]$$

where

$$\hat{S}(\zeta,s)=\int_0^\infty e^{-st}S^*(\zeta,t)\,dt. \qquad [77]$$

Now, $[74]_3$ requires

$$\hat{S}\text{ bounded as }\xi\to-\infty \qquad [78]$$

while $[74]_4$ gives:

$$\hat{S}(0,s)=\frac{1}{s+\epsilon^*}. \qquad [79]$$

The general solution of [76] is

$$\hat{S}(\zeta,s)=A(s)\,e^{(-\delta^*+\sqrt{\delta^{*2}+4s})\zeta/2}+B(s)\,e^{(-\delta^*-\sqrt{\delta^{*2}+4s})\zeta/2}\frac{+e^{-\beta^*\zeta}}{[s-(\beta^{*2}-\beta^*\delta^*)]}, \qquad [80]$$

where the boundedness condition [78] is guaranteed by the fact that

$$\frac{\alpha'}{\beta\left(\bar{S}+\dfrac{\gamma}{\beta}\right)}\equiv-\beta^*>0 \qquad [81]$$

provided, in addition, $B(s)\equiv 0$. It is recalled that it was shown in preceding section that the condition [81] holds automatically for $\rho_0<\rho_w$.

In order to satisfy [79], we need

$$A(s) = \frac{1}{(s+\epsilon^*)} - \frac{1}{[s-(\beta^{*2}-\beta^*\delta^*)]}, \qquad [82]$$

so that

$$\hat{S}(\zeta, s) = \left[\frac{1}{s+\epsilon^*} - \frac{1}{[s-(\beta^{*2}-\beta^*\delta^*)]}\right] e^{(-\delta^*/2+\sqrt{s+\delta^{*2}/4})\zeta} \; \frac{+e^{-\beta^*\zeta}}{[s-(\beta^{*2}-\beta^*\delta^*)]}. \qquad [83]$$

Now,

$$L_t^{-1}\left[\frac{1}{s+\epsilon^*} e^{\zeta\sqrt{s+\delta^{*2}/4}}\right] = \frac{e^{-\epsilon^* t}}{2}\left[e^{-\zeta\sqrt{\delta^{*2}/4-\epsilon^*}} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} + t^{1/2}\sqrt{\delta^{*2}/4-\epsilon^*}\right)\right.$$
$$\left. + e^{\zeta\sqrt{\delta^{*2}/4-\epsilon^*}} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} - t^{1/2}\sqrt{\delta^{*2}/4-\epsilon^*}\right)\right] \qquad [84]$$

and

$$L_t^{-1}\left[\frac{1}{[s-(\beta^{*2}-\beta^*\delta^*)]} e^{\zeta\sqrt{s+\delta^{*2}/4}}\right] = \frac{e^{(\beta^{*2}-\beta^*\delta^*)}}{2}\left[e^{-\zeta(\delta^{*2}/2-\beta^*)} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} + \left(\frac{\delta^2}{2}-\beta^*\right)t^{1/2}\right)\right.$$
$$\left. + e^{\zeta(\delta^*/2-\beta^*)} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} - \left(\frac{\delta^*}{2}-\beta^*\right)t^{1/2}\right)\right] \qquad [85]$$

while

$$L_t^{-1}\left[\frac{e^{-\beta^*\zeta}}{[s-(\beta^{*2}-\beta^*\delta^*)]}\right] = e^{-\beta^*\zeta+(\beta^{*2}-\beta^*\delta^*)t} \qquad [86]$$

Thus,

$$S^*(\zeta, t) = \frac{1}{2} e^{-\delta^*\zeta/2-\epsilon^* t}\left[e^{-\zeta\sqrt{\delta^{*2}/4-\epsilon^*}} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} + t^{1/2}\sqrt{\delta^{*2}/4-\epsilon^*}\right)\right.$$
$$\left. + e^{\zeta\sqrt{\delta^{*2}/4-\epsilon^*}} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} - t^{1/2}\sqrt{\delta^{*2}/4-\epsilon^*}\right)\right]$$
$$-\frac{1}{2} e^{(\beta^{*2}-\beta^*\delta^*)t}\left[e^{-\zeta(\delta^*-\beta^*)} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} + \left(\frac{\delta^*}{2}-\beta^*\right)t^{1/2}\right)\right.$$
$$\left. + e^{-\beta^*\zeta} \operatorname{Erfc}\left(-\frac{\zeta}{2t^{1/2}} - \left(\frac{\delta^*}{2}-\beta^*\right)t^{1/2}\right)\right]$$
$$+ e^{-\beta^*\zeta+(\beta^{*2}-\beta^*\delta^*)t} \equiv \Phi(\zeta, t), \; t > 0. \qquad [87]$$

Finally, an examination of the asymptotic behaviour of $S^*(\zeta, t)$ in [87] as $\zeta \to -\infty$ for finite t shows that, for condition $[68]_3$ to be satisfied, it is required that condition [81] hold together with

$$\delta^* \equiv \frac{\beta k(S_{o\max} - S_{o\min})(\rho_w - \rho_o) g(\mu_o + \mu_w)}{\phi(\mu_o - \mu_w)} \leq 0 \qquad [88]$$

In fact, the conditions $\rho_w > \rho_o$, $\beta < 0$ ensure that $\delta^* < 0$ while condition [81] has already been shown to be automatically satisfied for $\rho_w > \rho_o$.

The solution $S(\xi, \tau)$ of the original nonlinear initial value/boundary value problem is now obtained parametrically from [87] through the relations [75].

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